DEPARTMENT OF COMMERCE
LEWIS L. STRAUSS, Secretary

WEATHER BUREAU
F. W. REICHELDERFER, Chief

MONTHLY WEATHER REVIEW

JAMES E. CASKEY, JR., Editor

Volume 87
Number 4

APRIL 1959

Closed June 15, 1959
Issued July 15, 1959

THE EFFECTS OF VERTICAL VORTICITY ADVECTION AND TURNING OF THE VORTEX TUBES IN HEMISPHERIC FORECASTS WITH A TWO-LEVEL MODEL

G. 'ARNASON and L. P. CARSTENSEN, U.S. Weather Bureau [1]

Joint Numerical Weather Prediction Unit, Suitland, Md.

[Manuscript received December 10, 1958; revised March 9, 1959]

ABSTRACT

A 2-parameter forecasting model which includes the effects of vertical vorticity advection and turning of the vortex tubes is briefly described. Contributions of the above-mentioned terms are discussed and an example is presented.

The importance of consistent truncation over a large grid is pointed out.

1. INTRODUCTION

In the summer of 1957, the Joint Numerical Weather Prediction Unit (JNWP) changed from an IBM 701 to an IBM 704 electronic computer. Because of the higher capacity of the latter, the forecast area was extended to cover the Northern Hemisphere north of latitude 13°. Rather than recode the two-level, geostrophic model in use prior to the change of machine, it was decided to test a slightly more elaborate two-level model in the hope of improving the forecasts. This model was designed by Lt. Col. Philip D. Thompson, then Chief of the Research and Development Section of JNWP [1].

In testing this model some difficulties were encountered which had not been fully anticipated. These difficulties were due in part to attempted simple extension of the model from the one previously used, but more importantly to the increased importance of accumulated systematic small errors over a large grid.

This report is concerned mainly with consequences resulting from including the vertical advection of vorticity and the twisting terms in the prognostic equations.

[1] Present affiliation: Navy Numerical Weather Problems Group, U.S. Naval Postgraduate School, Monterey, Calif.

2. DESCRIPTION OF THE MODEL

The model's information levels are at 400 and 800 mb. As is customary, the ω-profile $\left(\omega=\frac{dp}{dt}\right)$ is assumed to have a smooth shape, and is for this particular model symmetrical in the vertical about $p=600$ mb. and zero for $p=200$ and $p=1000$ mb.

Application of the vorticity equation to the information levels results in two predictive equations:

$$\frac{\partial \eta_1}{\partial t}+\mathbf{v}_1\cdot\nabla\eta_1-\eta_1\frac{\partial\omega_1}{\partial p}+\omega_1\frac{\partial\eta_1}{\partial p}+\mathbf{k}\cdot\nabla\omega_1\times\frac{\partial\mathbf{v}_1}{\partial p}=0 \tag{1}$$

$$\frac{\partial \eta_2}{\partial t}+\mathbf{v}_2\cdot\nabla\eta_2-\eta_2\frac{\partial\omega_2}{\partial p}+\omega_2\frac{\partial\eta_2}{\partial p}+\mathbf{k}\cdot\nabla\omega_2\times\frac{\partial\mathbf{v}_2}{\partial p}=0 \tag{2}$$

where the subscripts 1 and 2 refer to 400 and 800 mb., respectively; notations are conventional.

We define the new variables

$$\eta^1=\frac{1}{2}(\eta_1-\eta_2);\ \mathbf{v}^1=\frac{1}{2}(\mathbf{v}_1-\mathbf{v}_2)$$
$$\bar{\eta}=\frac{1}{2}(\eta_1+\eta_2);\ \bar{\mathbf{v}}=\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2). \tag{3}$$

Addition and subtraction of (1) and (2), utilizing (3), result in

$$\frac{\partial\bar{\eta}}{\partial t}+\bar{\mathbf{v}}\cdot\nabla\bar{\eta}+\mathbf{v}^1\cdot\nabla\eta^1-\frac{1}{2}\left(\eta_1\frac{\partial\omega_1}{\partial p}+\eta_2\frac{\partial\omega_2}{\partial p}\right)+\frac{1}{2}\left(\omega_1\frac{\partial\eta_1}{\partial p}+\omega_2\frac{\partial\eta_2}{\partial p}\right)+\frac{1}{2}\mathbf{k}\cdot\left(\nabla\omega_1\times\frac{\partial\mathbf{v}_1}{\partial p}+\nabla\omega_2\times\frac{\partial\mathbf{v}_2}{\partial p}\right)=0$$

$$\frac{\partial\eta^1}{\partial t}+\bar{\mathbf{v}}\cdot\nabla\eta^1+\mathbf{v}^1\cdot\nabla\bar{\eta}-\frac{1}{2}\left(\eta_1\frac{\partial\omega_1}{\partial p}-\eta_2\frac{\partial\omega_2}{\partial p}\right)+\frac{1}{2}\left(\omega_1\frac{\partial\eta_1}{\partial p}-\omega_2\frac{\partial\eta_2}{\partial p}\right)+\frac{1}{2}\mathbf{k}\cdot\left(\nabla\omega_1\times\frac{\partial\mathbf{v}_1}{\partial p}-\nabla\omega_2\times\frac{\partial\mathbf{v}_2}{\partial p}\right)=0.$$

These equations are simplified when we observe that $\omega_1=\omega_2=A\omega_{600}$ where the proportionality factor A is assumed to be less than unity, and that the model can not distinguish between the p-derivatives of $\mathbf{v}$ and η at the two information levels. As a further step toward simplification, the derivatives with respect to p are replaced by finite differences as follows:

$$\frac{\partial\omega_1}{\partial p}=-\frac{\partial\omega_2}{\partial p}=\frac{\omega}{P};\quad \frac{\partial\eta_1}{\partial p}=\frac{\partial\eta_2}{\partial p}=-\frac{2\eta^1}{P};\quad \frac{\partial\mathbf{v}_1}{\partial p}=\frac{\partial\mathbf{v}_2}{\partial p}=-\frac{2\mathbf{v}^1}{P}\tag{4}$$

Where $P=400$ mb., and ω refers to the 600-mb. surface.

The prediction equations for $\bar{\eta}$ and η^1 now take the form:

$$\frac{\partial\bar{\eta}}{\partial t}+\bar{\mathbf{v}}\cdot\nabla\bar{\eta}+\mathbf{v}^1\cdot\nabla\eta^1-\frac{\omega}{P}\eta^1-\frac{2A\omega}{P}\eta^1-\frac{2A}{P}\mathbf{k}\cdot\nabla\omega\times\mathbf{v}^1=0\tag{5}$$

$$\frac{\partial\eta^1}{\partial t}+\bar{\mathbf{v}}\cdot\nabla\eta^1+\mathbf{v}^1\cdot\nabla\bar{\eta}-\frac{\omega\bar{\eta}}{P}=0.\tag{6}$$

It is pointed out that the fourth term in (5), resulting from the divergence terms of (1) and (2), is of the same form and magnitude as the fifth term in (5), which is the contribution of the vertical advection of vorticity. This implies that if the vertical advection and twisting terms were omitted from the vorticity equation, one should approximate the divergence term in such a way that similar terms are not introduced. As will appear later, this is of some importance for the vorticity balance.

It follows from the symmetry imposed upon the ω-distribution that the wind vector $\bar{\mathbf{v}}$ is divergence-free and can therefore be derived from a stream function $\bar{\psi}$. This is accomplished by solving the balance equation.

In deriving the differential wind and vorticity $\mathbf{v}^1$ and η^1, respectively, the geostrophic approximation is invoked, leading to the expressions:

$$\mathbf{v}^1=\frac{g}{2f}\mathbf{k}\times\nabla h;\quad \eta^1=\frac{g}{2f}\nabla^2 h\tag{7}$$

where h is the thickness of the layer 800–400 mb. Since

$$\bar{\mathbf{v}}=\mathbf{k}\times\nabla\bar{\psi}\quad\text{and}\quad\bar{\eta}=\nabla^2\bar{\psi}+f,\tag{8}$$

the two prediction equations (5) and (6) contain the three unknowns $\bar{\psi}$, h, and ω. The third equation needed for making the system of equations complete is the adiabatic equation, applied at 600 mb.,

$$\frac{\partial h}{\partial t}+\bar{\mathbf{v}}\cdot\nabla h-\frac{\sigma^2}{gP}\omega=0\tag{9}$$

where

$$\sigma^2=-\frac{p^2}{\rho\Theta}\frac{\partial\Theta}{\partial p}\tag{10}$$

and ρ is the density and Θ the potential temperature.

Utilizing the relations (7), (8), and (9), we may write the predictive equations (5) and (6) as follows:

$$\frac{\partial}{\partial t}\nabla^2\bar{\psi}+J(\bar{\psi},\bar{\eta})+\frac{g}{2f}J\left(h,\frac{g}{2f}\nabla^2 h\right)-\frac{1+2A}{2}\frac{g\omega}{fP}\nabla^2 h-\frac{gA}{fP}\nabla\omega\cdot\nabla h=0\tag{11}$$

$$\left(\nabla^2-\frac{2f\bar{\eta}}{\sigma^2}\right)\frac{\partial h}{\partial t}-\frac{2f\bar{\eta}}{\sigma^2}J(\bar{\psi},h)+fJ\left(\bar{\psi},\frac{1}{f}\nabla^2 h\right)+J(h,\bar{\eta})=0.\tag{12}$$

In solving the complete system (9), (11), and (12), one proceeds as follows:

1. solve equation (12) for $\partial h/\partial t$ and extrapolate in time;
2. compute ω from (9), using $\partial h/\partial t$ from step 1;
3. solve (11) for $\bar{\psi}$ at time $t+\Delta t$;
4. repeat the steps 1 to 3.

It can be disputed whether it is worth while including small terms like the twisting and vertical advection terms in such a crude model. As already mentioned, the model cannot distinguish between the vertical advections at the two information levels and the same applies to the twisting term.

In testing the model, however, it was decided to study the effects of the various terms in equation (11), and as a result three versions of (11) were dealt with; these are:

1. Equation (11) is replaced by

$$\frac{\partial}{\partial t}\nabla^2\bar{\psi}+J(\bar{\psi},\bar{\eta})-\frac{g\omega}{fP}\nabla^2 h=0\tag{13}$$

i.e., the third and fifth terms in (11) have been omitted and A has been set equal to ½;

2. Equation (11) is replaced by

$$\frac{\partial}{\partial t}\nabla^2\bar{\psi}+J(\bar{\psi},\bar{\eta})+\frac{g}{2f}J\left(h,\frac{g}{2f}\nabla^2 h\right)-\frac{2}{3}\frac{g\omega}{fP}\nabla^2 h=0\tag{14}$$

i.e., the twisting has been omitted;

3. Equation (11) is replaced by

$$\frac{\partial}{\partial t}\nabla^2\overline{\psi}+J(\overline{\psi},\overline{\eta})+\frac{g}{2f}J\left(h,\frac{g}{2f}\nabla^2 h\right)-\frac{g\omega}{fP}\nabla^2 h-\frac{g}{fP}\nabla\omega\cdot\nabla h=0 \quad (15)$$

i.e., the term $\frac{1}{2}\frac{g\omega}{fP}\nabla^2 h$, originating from the divergence terms in (1) and (2), has been neglected, and A set equal to unity. The latter is inconsistent with the assumption that A be less than unity, but this inconsistency is not of any significance.

3. COMMENTS ON VORTICITY BALANCE AND FICTITIOUS SOURCES OF VORTICITY

Omitting friction, we may write the vorticity equation as follows:

$$\frac{\partial\zeta}{\partial t}+\text{div}\,\eta\,\mathbf{v}+\mathbf{k}\cdot\nabla\times\omega\frac{\partial\mathbf{v}}{\partial p}=0. \quad (16)$$

Integration of (16) over an area A of a pressure surface bounded by a curve C gives

$$\frac{\partial\overline{\zeta}}{\partial t}+\frac{1}{A}\int_A \text{div}\,\eta\mathbf{v}dA+\frac{1}{A}\int_A\nabla\times\omega\frac{\partial\mathbf{v}}{\partial p}\cdot\mathbf{k}dA=\frac{\partial\overline{\zeta}}{\partial t}+\frac{1}{A}\int_C\eta\mathbf{v}\cdot\mathbf{n}dC+\frac{1}{A}\int_C\omega\frac{\partial\mathbf{v}}{\partial p}\cdot\mathbf{t}dC=0 \quad (17)$$

where dA is an area element, $\mathbf{n}$ a unit vector normal to the unit vector $\mathbf{t}$ which in turn is tangential to the curve C of which dC is an element; $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{k}$ form a right-hand system.

We substitute for $\mathbf{v}$ from

$$\mathbf{v}=\mathbf{k}\times\nabla\psi+\nabla X \quad (18)$$

into (17) which takes the form

$$\frac{\partial\overline{\zeta}}{\partial t}+\frac{1}{A}\left[\int_C\eta d\psi+\int_C\eta\nabla X\cdot\mathbf{n}ds+\int_C\omega\frac{\partial\mathbf{v}}{\partial p}\cdot d\mathbf{r}\right]=0 \quad (19)$$

Thus the mean vorticity $\overline{\zeta}$ may be changed as follows:

(a) by advection of vorticity across the lateral boundaries by the divergence-free part of the wind;

(b) by vorticity advection across the lateral boundaries by the non-rotational part of the wind and due to its divergence ($\nabla X\cdot\nabla\eta$ and $\eta\text{div}\mathbf{v}$);

(c) vertical advection of circulation along the boundary curve C.

If we impose the boundary condition $\mathbf{v}\cdot\mathbf{n}=0$ and $\omega=0$, i.e., if the system is closed, the mean vorticity is independent of time. Hence, as long as the vorticity equation is not approximated there are no vorticity sources. This is of interest to bear in mind when dealing with simple models where approximations are inevitable. For example, in omitting the advection of vorticity by the non-rotational part of the wind in the vorticity equation and keeping the divergence term, a false vorticity source may thereby be introduced, upsetting the vorticity balance. This argument applies also to the twisting and vertical advection terms in the vorticity equation; retaining one of them and omitting the other introduces a spurious vorticity source.

As will be shown later, this introduction of spurious vorticity sources is not to be taken lightly when dealing with forecasts over a large area. For a hemispheric grid it introduces a large-scale error devastating for the usefulness of the forecast.

4. DISCUSSION OF THE FORECASTS

All forecasts discussed here were made using the JNWP octagonal grid which covers the Northern Hemisphere north of latitude 13°.

In investigating the performance of the models, the contributions of certain of the terms of the prediction equation for $\overline{\psi}$ were computed separately and then accumulated for 12-hour periods.

1. Forecasts using equation (13) were carried to 48 hours for 4 cases with results which were similar with respect to the features of interest here. In these four cases the contribution of the third term of (13) was isolated. The forecast with initial time 1500 GMT, April 3, 1958, may be taken as representative of this group, and is reproduced as figure 1. During the first 12 hours the vertical advection of vorticity contributed a negative height change [2] everywhere with maximum values of −600 feet. During the second 12-hour period an area of positive changes up to 160 feet appeared, but decreased negative changes persisted over most of the map. During the third 12-hour period some small negative changes persisted, but the positive contributions greatly enlarged with maximum values up to 370 feet. During the last 12-hour period the contributions were everywhere positive with maximum values of 400 feet. The contribution of this term appeared to be mainly large-scale with little detail on the scale of the principal synoptic features.

The phase relationships between h and $\overline{\psi}$ and between h and w are illustrated in figure 2. The vertical velocity field, initially as well as throughout the 48-hour forecast period, was well ahead of the streamline field (upward motion ahead of a trough). On the other hand, the thickness field started out lagging behind the stream function, caught up at around 24 hours, and was well ahead at 48

[2] The quantity ψ occurring in the figures has the dimension length and is derived from $\overline{\psi}$ through the relation $\overline{\psi}=\frac{g}{f_o}\psi$ where g is the acceleration of gravity and f_o is the Coriolis parameter at 45° latitude. In the captions to the figures, ψ is alternatively referred to as height and stream function.

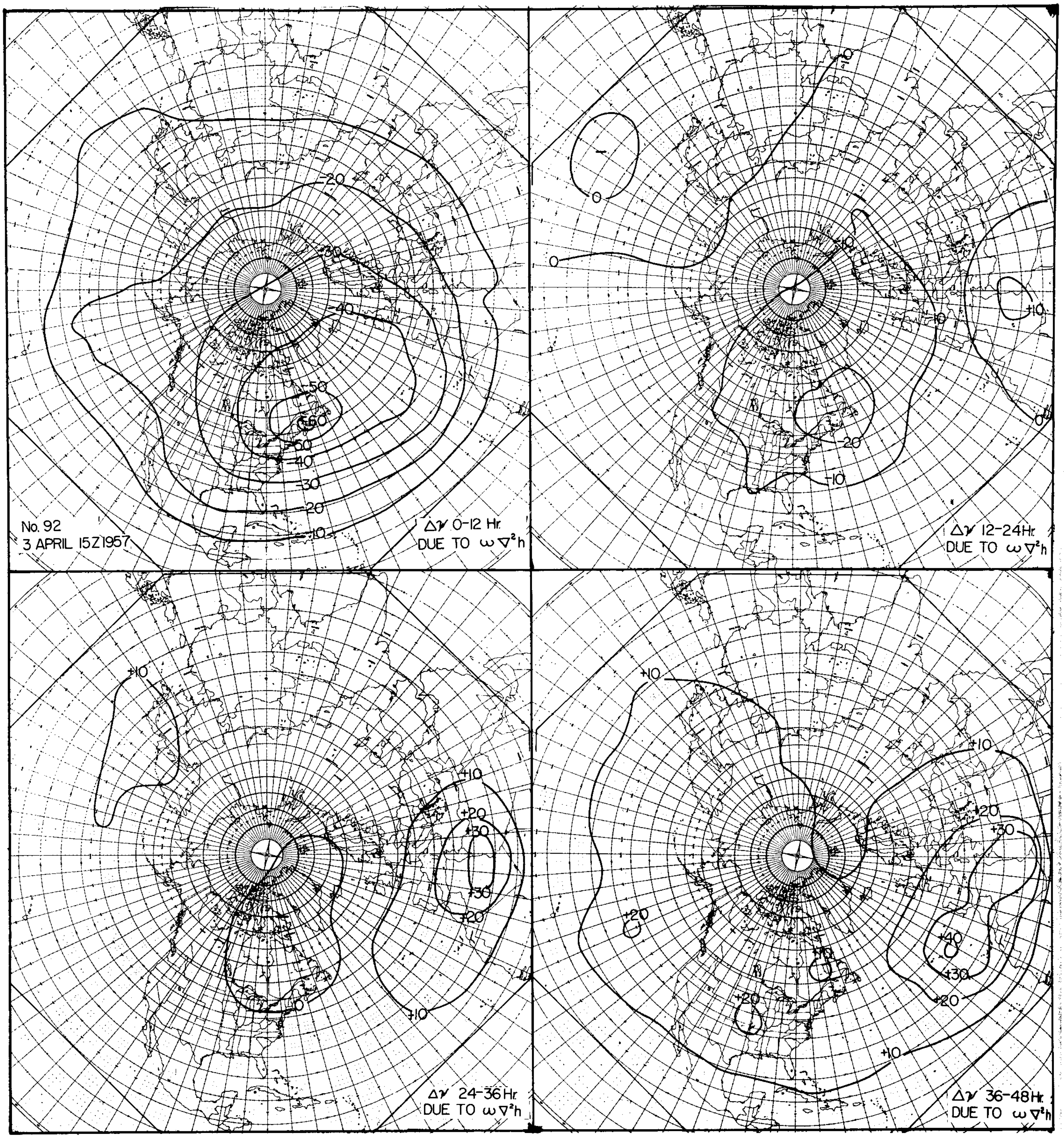

Figure 1.—Change in 500-mb. height due to vertical advection of vorticity, labeled in tens of feet.

hours. The error maps show that the stream function field was moved too slowly, and indicate too fast a movement of the thickness field. The w-field versus h-field shows that w and h were positively correlated initially and negatively correlated after 24 hours. This would seem to explain the reversal in the vertical advection of vorticity as time increased.

2. A forecast using (14) was carried to 48 hours, again using as initial data 1500 GMT, April 3, 1958.

In this forecast the thickness field kept its lag better than with the previous model, but during the 36–48-hour period it caught up with the stream function field in

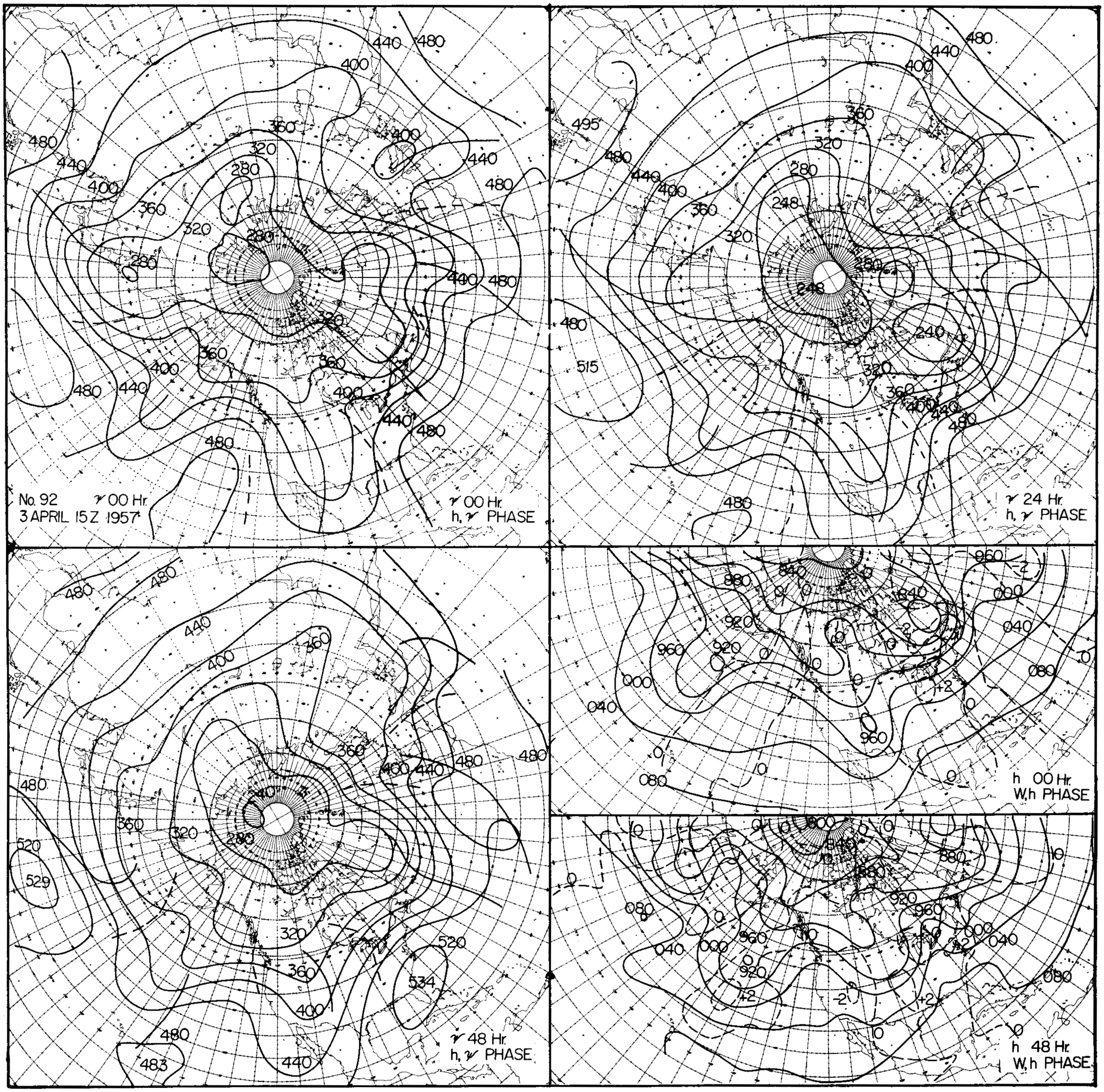

FIGURE 2.—Phase lags between thickness and stream function and thickness and vertical velocity. In portraying the h, ψ phase, dashed lines indicate ridges and short solid lines troughs in the h-field (not contoured); the labeled solid lines are ψ-contours. In the two maps showing the phase between thickness and vertical velocity (h,w), solid lines are thickness contours (in tens of feet) and dashed lines contours for vertical velocity in cm. sec.$^{-1}$.

parts of the map. The vertical advection contribution is shown in figure 3. There was no reversal in the contribution of the vertical advection of vorticity term, but the negative contribution to stream function change was less during the 24–48-hour periods than during the first 24 hours. Vertical velocity had much the same phase relationship to stream function as before, but there is considerable difference in detail between the fields. The 24-hour vertical velocity with (14) was more intense, and the 48-hour vertical velocity slightly more intense than results from using (13).

3. Six forecasts were carried out using equation (15) as one of the basic prognostic equations. Here again the results were similar with respect to the features under

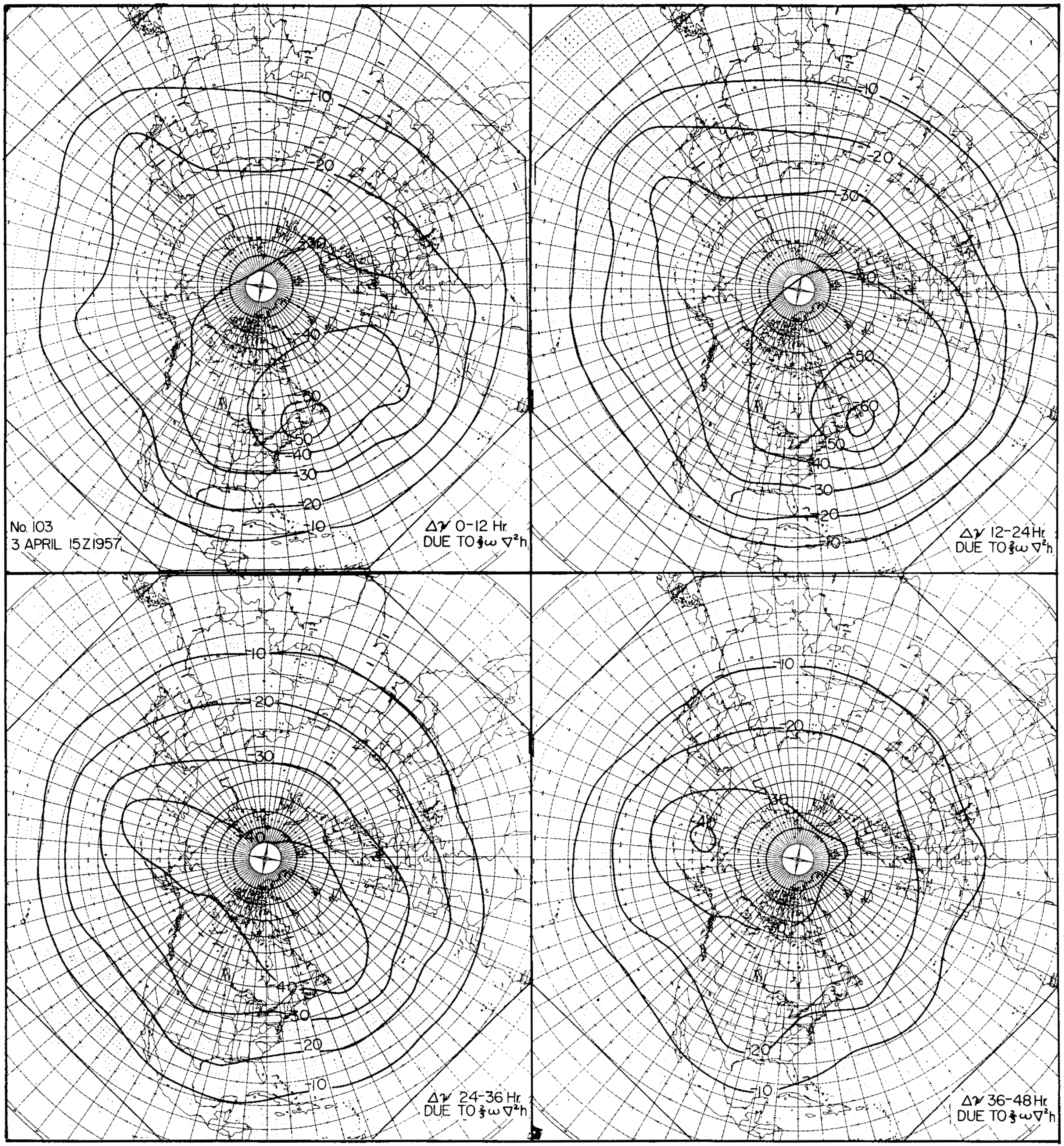

Figure 3.—Change in 500-mb. height due to vertical advection of vorticity, labeled in tens of feet.

consideration, and the case of 1500 GMT, April 3, 1958, may be taken as typical. In this case the last two terms of equation (15) were accumulated separately for each 12 hours, the results being given in figure 4. It is seen that the integrated values almost everywhere have opposite signs and tended to cancel each other. The vertical advection term was the larger of the two but this is mainly because of different degrees of truncation. In the vertical advection term, the Laplacian of thickness is formed using the mesh length of d, whereas both of the

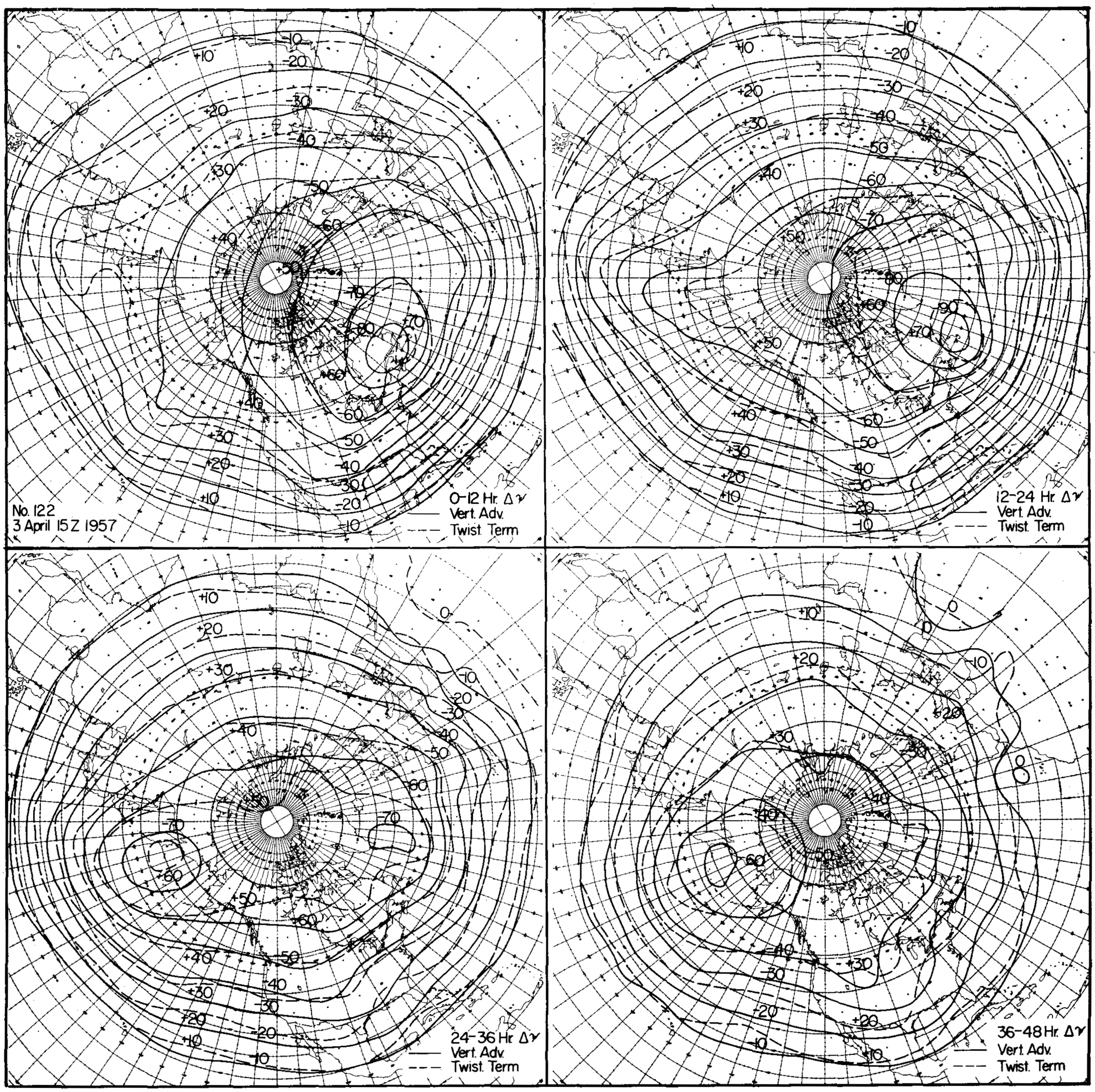

FIGURE 4.—Change in 500-mb. height due to vertical advection (solid lines) and twisting (dashed lines) terms.

first derivatives in the twisting terms have $2d$ as mesh length. When the inconsistency of truncation is removed by combining the two terms it becomes clear, as pointed out earlier, that the net contribution over the grid of the two terms depends only on conditions at the lateral boundaries.

When the last two terms of (15) are combined before finite differences are taken, the accumulated sums take on the values given in figure 5. These may be compared with the results shown in figure 6, which is the sum of the twisting and vertical advection terms of figure 4. These results should be the same except for truncation errors. It thus appears that on a large grid inconsistent truncation of small but systematic terms may alone account for significant forecast error.

It is also to be remarked that in this model and information grid the vertical velocity terms contributed very little in comparison with the error remaining. However, it would be unwise to conclude from this experiment alone that the terms are really unimportant either to the atmosphere or to more sophisticated models.

This model, when corrected for the systematic error

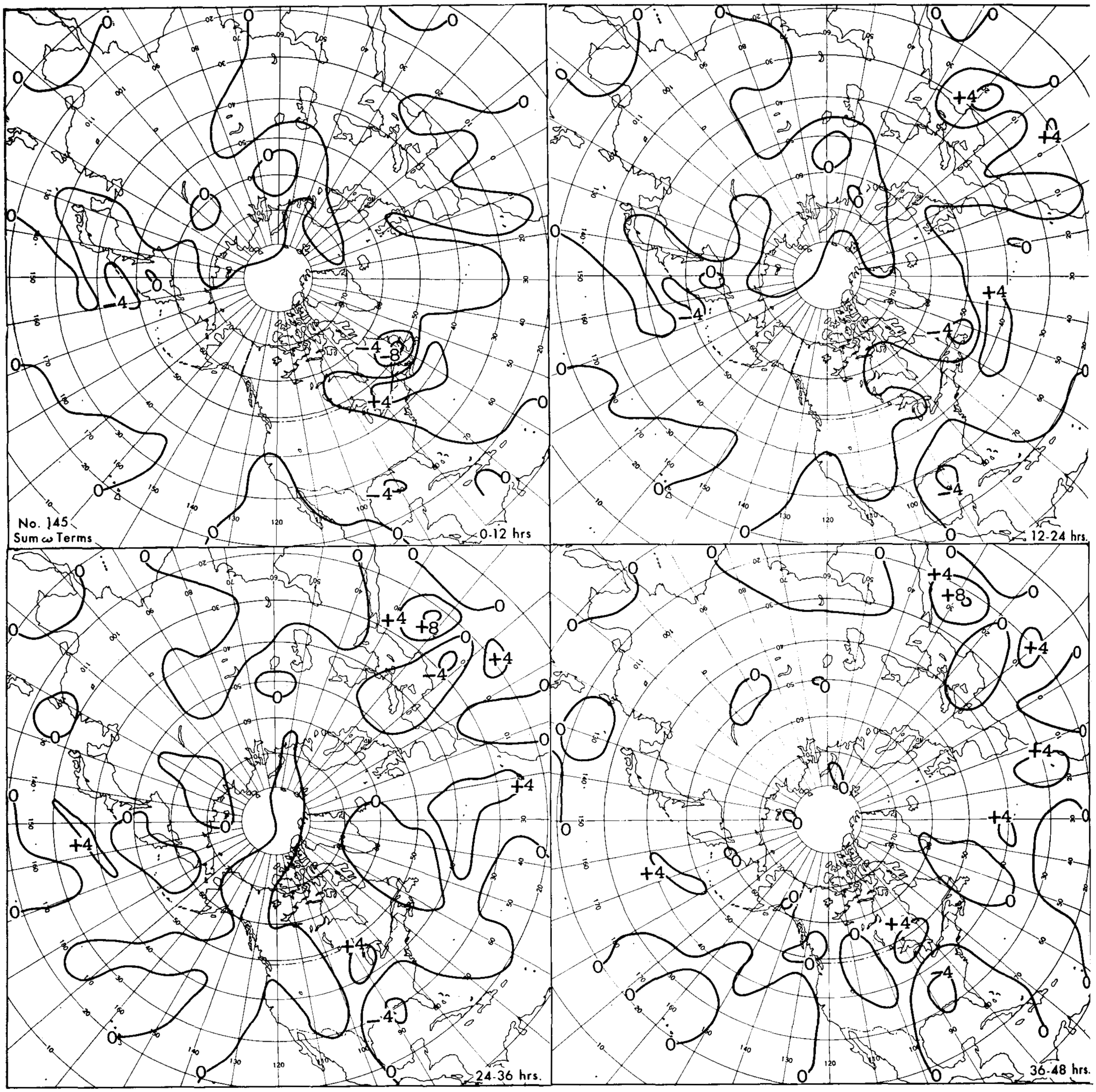

Figure 5.—Change in 500-mb. height due to combined vertical advection and twisting terms consistently truncated.

described above, showed a tendency to move the subtropical Highs erroneously westward, and a tendency toward overdevelopment of Lows. The spurious westward movement is presumably a manifestation of this model's incapability of properly forecasting the long waves as discussed by Wolff [2] and Cressman [3]. The overdevelopment is attributed to the third term in equations (14) and (15). This term would seem to be incapable of properly accounting for observed genesis of mean vorticity $\bar{\zeta}$.

5. CONCLUSIONS

The conclusions reached in this study may be summarized as follows:

1. Each of the vertical advection and twisting terms in the vorticity equation tends to have a uniform sign over the entire grid. In a closed system their net contribution is, however, zero when taken together. Omitting one and retaining the other of these two terms leads to spurious creation or destruction of vorticity, devastating to a hemispheric forecast.

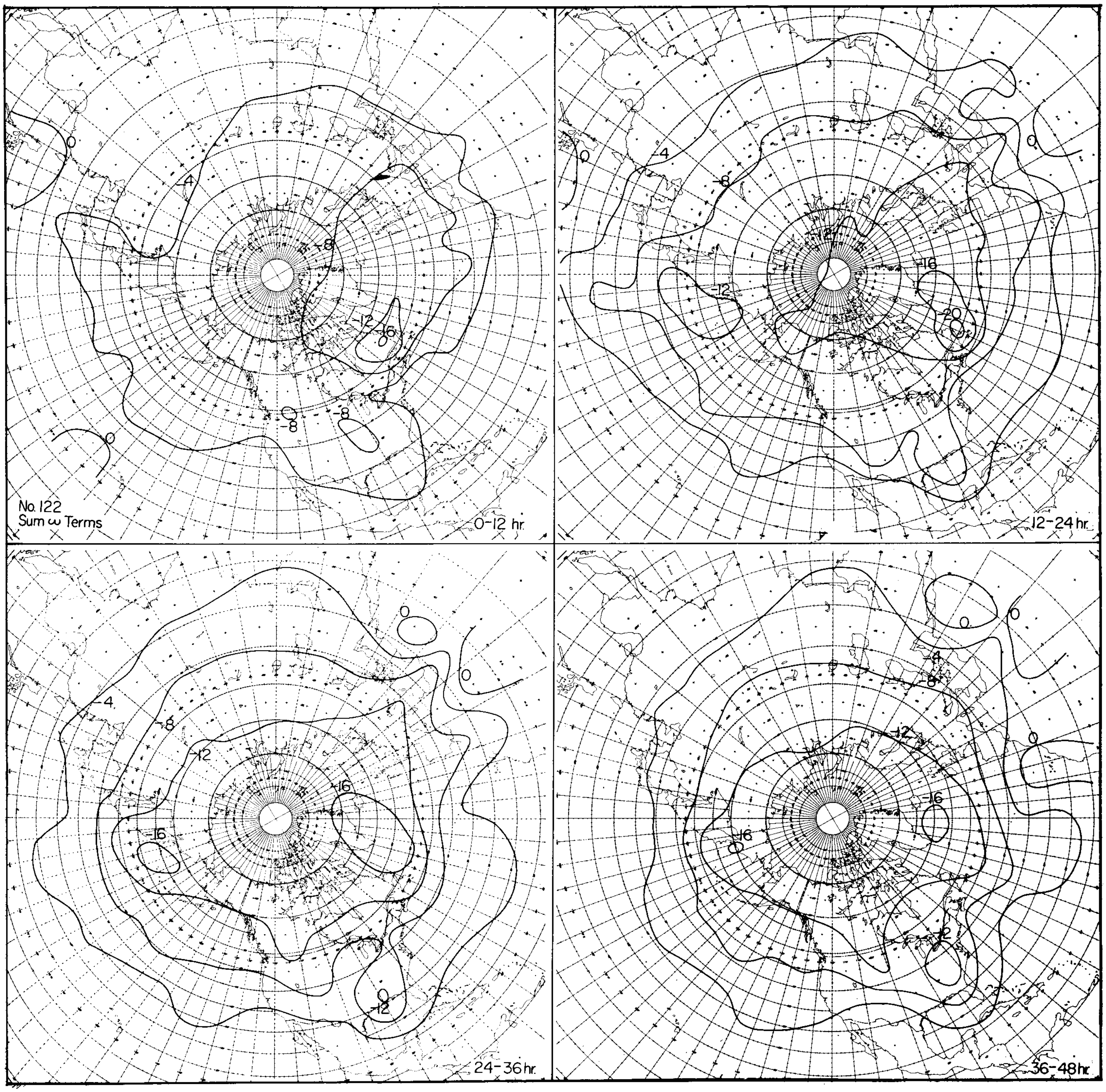

FIGURE 6.—Change in 500-mb. height due to vertical advection and twisting terms inconsistently truncated.

2. When both terms are included, inconsistent truncation may lead to considerable error in the forecast.

3. In approximating the divergence term in the vorticity equation, precaution should be taken not to introduce a small but systematic error of the type referred to under conclusion 1.

4. The contribution of the vertical advection and twisting terms when taken together was insignificant in this model in the six cases tested.

508099—59——2

REFERENCES

1. P. D. Thompson, "A Two-Level Model with Effects of Vertical Velocity Advection, Irregular Terrain, and Variable Static Stability," Office Note No. 6, Joint Numerical Weather Prediction Unit, June 1957.
2. P. M. Wolff, "The Error in Numerical Forecasts Due to Retrogression of Ultra-Long Waves," *Monthly Weather Review*, vol. 86, No. 7, July 1958, pp. 245–250.
3. George P. Cressman, "Barotropic Divergence and Very Long Atmospheric Waves," *Monthly Weather Review*, vol. 86, No. 8, August 1958, pp. 293–297.